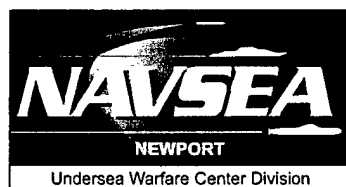# Derivation of Beam Interpolation Coefficients with Application to the $K-\omega$ Beamformer

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PREFACE

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A problem of general interest is to be able to use a set of L beams, steered and focused respectively at the bearing-range pairs $[\theta_1, R_1], [\theta_2, R_2], \cdots, [\theta_L, R_L]$, to approximate a beam steered and focused at a given angle-range pair $[\theta_0, R_0]$. A problem of particular interest is to be able to use L beams focused at infinite range (i.e., a plane wave) to approximate the beam steered and focused at the angle-range pair $[\theta_0, R_0]$. To accomplish the latter, a set of L appropriate coefficients, which are complex in general, must be computed and then used to scale and combine the beams to generate an approximation of the beam in question. These coefficients are derived in this report via mean square error minimization. The number of coefficients needed for a given error value is examined as a function of frequency and focusing range. The beam interpolation scheme is applied to the K–ω line array.

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1. INTRODUCTION

A problem of general interest is to be able to use a set of L beams, steered and focused respectively at the bearing-range pairs $[\theta_1, R_1], [\theta_2, R_2], \cdots, [\theta_L, R_L]$ to approximate a beam steered and focused at a given angle-range pair $[\theta_0, R_0]$. A problem of particular interest is to be able to use L beams focused at infinite range (plane wave) to approximate the beam steered and focused at the angle-range pair $[\theta_0, R_0]$.

To do this, a set of L appropriate coefficients, which are complex in general, must be computed and then used to scale and combine the beams to generate an approximation of the beam in question. These coefficients are derived in this report via mean square error (MSE) minimization. Also, the number of coefficients needed for a given error value is examined as a function of frequency and focusing range. The beam interpolation scheme is applied to the K-ω beamformer.

Throughout, vectors are denoted by bold, lowercase letters and matrices by bold, uppercase bold letters. All others are scalars. The complex notation j (where $j = \sqrt{-1}$) is used. The superscript $(^{\mathrm{T}})$ is used to denote vector and matrix transpose.

2. GENERAL OBSERVATIONS

Almost certainly, the steering angle θ_0 of the approximated beam must lie inside the bearing span of approximating beams $[\theta_1, \theta_L]$.

The choice of beam bearing spacing for the set of reference beams is critical for the proper formation of the desired beam steered/focused at a given bearing/range. If the reference beams are spaced too far apart in bearing, then information is lost and the desired beam will not be properly formed. If, on the other hand, the reference beams are spaced too closely together in bearing, then the information provided by each beam becomes highly correlated. This usually results in large positive and negative complex coefficients, leading to numerical instability.

The ideal spacing for a given frequency occurs when the beam responses are orthogonal. For a fixed beam spacing this condition can be satisfied for only one frequency. For the other frequencies, the reference beam set will either be under-sampled or over-sampled. For the case of a K-ω beamformer, the beams in K (bearing) space are always orthogonal for all frequencies, which follows from the Fourier transform property. This makes the K-ω beamformer an ideal choice for implementing this technique. This is especially true when the reference beam set is formed at infinite range.

3. COEFFICIENT DERIVATION

3.1 FORMULATION OF THE MEAN SQUARE ERROR (MSE)

A steered and focused narrowband beampattern for an N-channel line array is given by

$$\sum_{i=1}^{N} w_i e^{j2\pi f[\tau_i(\theta,R)-\tau_i(\theta_0,R_0)]}.\tag{1}$$

The time delay $\tau_i(\theta, R)$ is the time required for a spherical wavefront of radius R to travel at a speed c from the center of the array (at coordinate 0) to element i (at coordinate x_i). Denoting the distance between the wavefront source to element i by R_i (see figure 1), one has

$$\tau_i(\theta, R) = \frac{R - R_i}{c}.$$
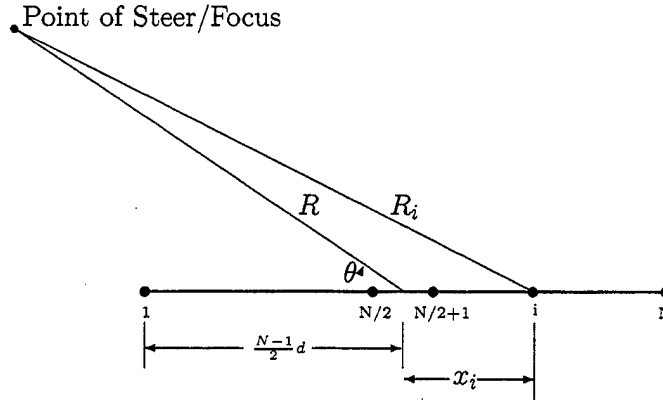
Figure 1. Time Delay Computation for a Curved Wavefront

By using the law of cosines, R_i can be expressed as a function of x_i and the source range R. Completing the square and factoring $(R - x_i \cos\theta)$ out, the time delay becomes

$$\tau_i(\theta, R) = \frac{1}{c}\left[R - (R - x_i \cos\theta)\sqrt{1 + \left(\frac{x_i \sin\theta}{R - x_i \cos\theta}\right)^2} \right],$$

$$\approx \underbrace{\frac{x_i \cos\theta}{c}}_{\text{plane}} - \underbrace{\frac{x_i^2 \sin^2\theta}{2c(R - x_i \cos\theta)}}_{\text{curvature}}, \qquad R \gg x_i \tag{2}$$

where

$\tau_i(\theta, R)$ = Target time delay for a target at bearing θ and range R,

$\tau_i(\theta_0, R_0)$ = Beamformer time delay for a steering angle θ_0 and focus range R_0,

x_i = Coordinate of element i with respect to a given reference,

c = Speed of sound (beamformer and actual speeds of sound are assumed equal),

f = Target frequency, and

w_i = Shading weight applied to channel i, normalized so that $\sum w_i = 1$.

The approximation in equation (2) is the basic $\sqrt{1+a} \approx 1 + \frac{a}{2}$ for $a \ll 1$, which is true in this case for ranges large enough when compared to array dimensions.

The generally complex coefficients that will be used to combine the L-beams to approximate a single beam, can be derived by minimizing the MSE between the actual beam desired, steered to θ_0 and focused at R_0, and the weighted sum of L beams, steered to $[\theta_1, \theta_2, \cdots, \theta_L]$ and focused at $[R_1, R_2, \cdots, R_L]$, respectively. However, to simplify derivation and implementation, minimization of the sum over elements of the individual MSEs for the respective actual versus interpolated beams will instead be performed. Denoting the interpolation coefficients by $[\alpha_1 + j\beta_1, \alpha_2 + j\beta_2, \cdots, \alpha_L + j\beta_L]$, the sum over elements of the individual MSEs, which must be minimized over the coefficient $\alpha_p + j\beta_p$, can be expressed by

$$E = \sum_{i=1}^{N} \left| w_i e^{j2\pi f[\tau_i(\theta, R) - \tau_i(\theta_0, R_0)]} - \sum_{p=1}^{L} (\alpha_p + j\beta_p) e^{j2\pi f[\tau_i(\theta, R) - \tau_i(\theta_p, R_p)]} \right|^2. \qquad (3)$$

It is assumed in equation (3) that the sensor elements of the L beams –which are weighted and summed to produce the new beam– are unshaded and, hence, the resultant coefficients will include (i.e., perform), the shading, in addition to the focusing-steering operations. For the case of obtaining coefficients that do not include shading, a slighly different vector/matrix definition must be used in the solution equation (9), and this definition is given in appendix A.

Factoring out $e^{j2\pi f[\tau_i(\theta, R) - \tau_i(\theta_0, R_0)]}$ from the inner terms of equation (3) and replacing the magnitude operator by the equivalent product of the inner quantity and its conjugate, and using the substitution $v_i(p) = 2\pi f[\tau_i(\theta_0, R_0) - \tau_i(\theta_p, R_p)]$, one obtains

$$
\begin{aligned}
E &= \sum_{i=1}^{N} \left[w_i - \sum_{p=1}^{L} (\alpha_p + j\beta_p) e^{jv_i(p)} \right] \left[w_i - \sum_{q=1}^{L} (\alpha_q - j\beta_q) e^{-jv_i(q)} \right], \\
&= \sum_{i=1}^{N} \left[w_i^2 - w_i \sum_{q=1}^{L} (\alpha_q - j\beta_q) e^{-jv_i(q)} - w_i \sum_{p=1}^{L} (\alpha_p + j\beta_p) e^{jv_i(p)} \right. \\
&\qquad \left. + \sum_{p=1}^{L} \sum_{q=1}^{L} (\alpha_p + j\beta_p)(\alpha_q - j\beta_q) e^{j[v_i(p) - v_i(q)]} \right].
\end{aligned}
\qquad (4)
$$

6

Define the $L \times 1$ vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\mathbf{c}$, and $\mathbf{s}$ as follows:

$$
\begin{aligned}
\boldsymbol{\alpha} &= [\alpha_1, \alpha_2, \cdots, \alpha_L]^{\mathrm{T}}, \\
\boldsymbol{\beta} &= [\beta_1, \beta_2, \cdots, \beta_L]^{\mathrm{T}}, \\
\mathbf{c} &= \left[\sum_{i=1}^{N} w_i \cos v_i(1), \sum_{i=1}^{N} w_i \cos v_i(2), \cdots, \sum_{i=1}^{N} w_i \cos v_i(L)\right]^{\mathrm{T}}, \\
\mathbf{s} &= \left[\sum_{i=1}^{N} w_i \sin v_i(1), \sum_{i=1}^{N} w_i \sin v_i(2), \cdots, \sum_{i=1}^{N} w_i \sin v_i(L)\right]^{\mathrm{T}}.
\end{aligned}
\tag{5}
$$

In addition, define the $L \times L$ matrices $\mathbf{C}$ and $\mathbf{S}$ so that

$$
\begin{aligned}
[\mathbf{C}]_{pq} &= \sum_{i=1}^{N} \cos[v_i(q) - v_i(p)] = [\mathbf{C}]_{qp}, \\
[\mathbf{S}]_{pq} &= \sum_{i=1}^{N} \sin[v_i(q) - v_i(p)] = -[\mathbf{S}]_{qp}.
\end{aligned}
\tag{6}
$$

Also, for general $L \times 1$ vectors $\mathbf{a}$, $\mathbf{b}$ with p^{th} component notations $[\mathbf{a}]_p$ and $[\mathbf{b}]_p$, respectively, and the $L \times L$ matrix $\mathbf{A}$ with pq^{th} component notation $\mathbf{A}_{pq}$, one has

$$
\sum_{p=1}^{L} [\mathbf{a}]_p [\mathbf{b}]_p = \mathbf{a}^{\mathrm{T}} \mathbf{b},
$$

and

$$
\sum_{p=1}^{L} \sum_{q=1}^{L} [\mathbf{a}]_p [\mathbf{b}]_q [\mathbf{A}]_{pq} = \mathbf{a}^{\mathrm{T}} \mathbf{A} \mathbf{b}.
$$

Finally, using the Euler identity $e^{j\gamma} = \cos\gamma + j \sin\gamma$ to convert all complex exponentials to cosines and sines, the scalar E in equation (4) can be expressed in vector notation as

$$
E = \sum_{i=1}^{N} w_i^2 - 2\boldsymbol{\alpha}^{\mathrm{T}}\mathbf{c} + 2\boldsymbol{\beta}^{\mathrm{T}}\mathbf{s} + \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{C}\boldsymbol{\alpha} + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{C}\boldsymbol{\beta} - \boldsymbol{\alpha}^{\mathrm{T}}\mathbf{S}\boldsymbol{\beta} + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{S}\boldsymbol{\alpha}.
\tag{7}
$$

3.2 MINIMIZATION OF MSE AND DERIVATION OF COEFFICIENTS

To minimize E as a function of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, the derivative of E with respect to $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ is computed and set to zero. The vector and matrix gradients identities of Scharf [1], which will be used in minimization, are repeated here for convenience:

$$
\frac{\partial}{\partial \mathbf{x}} \mathbf{b}^{\mathrm{T}} \mathbf{x} = \frac{\partial}{\partial \mathbf{x}} \mathbf{x}^{\mathrm{T}} \mathbf{b} = \mathbf{b},
$$

and

$$
\frac{\partial}{\partial \mathbf{x}} \mathbf{x}^{\mathrm{T}} \mathbf{A} \mathbf{x} = 2\mathbf{A}\mathbf{x},
$$

where $\mathbf{A}$, an $L \times L$ matrix, and $\mathbf{b}$, an $L \times 1$ vector, are independent of $\mathbf{x}$, an $L \times 1$ vector. Setting $\frac{\partial}{\partial \boldsymbol{\alpha}} E = 0$ and $\frac{\partial}{\partial \boldsymbol{\beta}} E = 0$ gives

$$\begin{aligned}
\mathbf{C}\alpha - \mathbf{S}\beta &= \mathbf{c}, \\
\mathbf{S}\alpha + \mathbf{C}\beta &= -\mathbf{s}.
\end{aligned} \tag{8}$$

Solving equations (8) for α and β, one gets

$$\boxed{\begin{aligned}
\alpha &= (\mathbf{C} + \mathbf{S}\mathbf{C}^{-1}\mathbf{S})^{-1}(\mathbf{c} - \mathbf{S}\mathbf{C}^{-1}\mathbf{s}), \\
\beta &= -\mathbf{C}^{-1}(\mathbf{s} + \mathbf{S}\alpha),
\end{aligned}} \tag{9}$$

and the coefficients are the $L \times 1$ complex vector $\alpha + j\beta$.

4. APPLICATION TO LINE ARRAY K-ω

4.1 K-ω BACKGROUND

The digital K-ω, or wavenumber-frequency, field is generated by way of a two-dimensional Fast Fourier Transform (FFT) over time and space. The temporal FFT is applied on an N_{fft}-block of the sampled channel output, where N_{fft} is a radix-two number. Usually, the method of overlap-and-save is used for the time-series data. The spatial FFT, however, is applied along elements, where zero-padding is usually used to increase the number of elements N to the radix-two number M_{fft} for FFT application. Care must be taken, however, when computing the matrices $\mathbf{C}$ and $\mathbf{S}$ in equation (6) after zero-padding the array elements, to set the matrix entries corresponding to the padded zeros to zero.

The relationship between K-ω beamforming, which is the two-dimensionald discrete fourier transform (DFT) over space (elements) and time (element outputs), and time-delay beamforming arises from the similarity between the types of beamforming and the DFT equations. For a line array with N elements indexed i, equi-spaced at distance d, and shaded by w_i, the planewave beamformer output for an incoming source signal $x_i(t)$ measured at element i is expressed by

$$\sum_{i=1}^{N} w_i x_i[t - \tau_i(\theta)], \tag{10}$$

where the beamformer time-delay for a planewave steered to angle θ is

$$\tau_i(\theta) = \frac{1}{c} \underbrace{\left(\frac{N+1}{2} - i \right) d \cos\theta}_{x_i}, \quad i = 1, 2, \cdots, N. \tag{11}$$

Applying, first, the temporal Fourier transform to the beamformer in equation (10), substituting for $\tau_i(\theta)$ in equation (11), and shifting the index i from the range $[1, N]$ to $[0, N-1]$ yields

$$e^{-j2\pi f \frac{N-1}{2} \frac{d}{c} \cos\theta} \sum_{i=0}^{N-1} w_i X_i(f) e^{j2\pi f \frac{d}{c} i \cos\theta}, \tag{12}$$

where f is the frequency variable and $X_i(f)$ is the temporal Fourier transform of the signal $x_i(t)$.

To implement the sum over elements in equation (12) using the spatial M_{fft}-length FFT, with $M_{\text{fft}} - N$ zero padding, the following change of variables must be performed:

$$f \frac{d}{c} \cos\theta = \frac{k}{M_{\text{fft}}}, \Longrightarrow$$

$$k = \frac{M_{\text{fft}} d \cos\theta}{c} f, \tag{13}$$

which, for a target at the array design frequency ($\lambda = 2d = c/f$), or any other smaller frequency, will lead to a $k \in [-\frac{M_{\text{fft}}}{2}, \frac{M_{\text{fft}}}{2}]$. Negative values for k indicate $\theta > 90°$, while positive ones indicate $\theta < 90°$; $k = 0$ indicates a broadside target. Since, however, the FFT considers the integer range $[0, M_{\text{fft}} - 1]$, the $\theta > 90°$ values will appear in the range $[M_{\text{fft}/2}, M_{\text{fft}}]$, while the $\theta \leq 90°$ values will appear in the range $[0, M_{\text{fft}/2} - 1]$.

Going back to the FFT implementation of K-ω, after substituting N with the radix-2 M_{fft}, $w_i X_i(f)$ with $y(i)$ for convenience, and the values of k from equation (13), equation 12 representing the K-ω field $X(k, f)$ becomes

$$X(k, f) = e^{-j\pi \frac{M_{\text{fft}}-1}{M_{\text{fft}}} k} \sum_{i=0}^{M_{\text{fft}}-1} y(i) e^{j2\pi \frac{ik}{M_{\text{fft}}}}. \tag{14}$$

To implement equation (14) via a forward FFT, the index i must first be flipped from $[0, M_{\text{fft}} - 1]$ to $[-(M_{\text{fft}} - 1), 0]$. Then, performing the index substitution $i \rightarrow i - (M_{\text{fft}} - 1)$ and factoring the non-index-dependent phase out of the summation, equation (14) becomes

$$X(k, f) = e^{j\pi \frac{M_{\text{fft}}-1}{M_{\text{fft}}} k} \sum_{i=0}^{M_{\text{fft}}-1} y(M_{\text{fft}} - 1 - i) e^{-j2\pi \frac{ik}{M_{\text{fft}}}}, \tag{15}$$

which may be implemented via a forward FFT. The term $y(M_{\text{fft}} - 1 - i)$ represents a flipped version of $y(i)$, which is easily obtained by applying the spatial FFT with the element order reversed.

For the frequency axis, the usual bin-to-frequency formula is used:

$$f = \frac{l}{N_{\text{fft}}} F_s,$$

where f is the frequency of interest corresponding to bin i, F_s is the temporal sampling frequency, N_{fft} is the FFT length, and $l = 0, 1, \cdots, N_{\text{fft}} - 1$ is the frequency bin index.

Each point of the K-ω field defined by its coordinates, frequency bin $= l$ and wavenumber bin $= k$, represents a frequency filter that allows targets at frequency corresponding to bin i and a spatial filter that allows targets at bearing obtained by substituting k in equation (13).

The location of the spatial filter is a function of the frequency of interest. For different frequencies $f = c/\lambda$, a given maximum response axis (MRA) corresponds to different k values defined by equation (13). More specifically, equation (13) prescribes a linear relationship between the frequency (c/λ) and wavenumber k. Thus, a broadband target with bearing θ appears in the K-ω spectrum on a straight line defined by

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$$k = \begin{cases} \frac{M_{\text{fft}}\, d \cos\theta}{c} f & 0° \leq \theta \leq 90° & (k = 0, 1, \cdots, M_{\text{fft}}/2 - 1), \\ \frac{M_{\text{fft}}\, d \cos\theta}{c} f + M_{\text{fft}} & 90° < \theta \leq 180° & (k = M_{\text{fft}}/2, M_{\text{fft}}/2 + 1, \cdots, M_{\text{fft}} - 1), \end{cases} \tag{16}$$

where f is the frequency variable.

4.2 FOCUS RANGE VERSUS MRA FORMULA

When designing different sets of focus ranges, it is of interest to keep the scalloping loss between neighboring range sets fixed. The width of the aperture used for focusing determines its range sensitivity. Since the effective width of the aperture seen by a target located at bearing θ is a function of $\sin\theta$, and since a curvature across an aperture (i.e., range sensitivity) varies as a function of the square of the aperture width, a focusing range that varies as a function of $\sin^2(\text{MRA})$, and hence k, must be devised.

Furthermore, provided that the quantity $R - x_i \cos\theta$ in equation (2) is not significantly different from R (i.e., $R \gg x_i \cos\theta$), which is true at not-so-oblique bearings, then the time delay can be viewed approximately as a linear function of inverse range – a fact that must also be used in designing focusing range locations. Thus, the scalloping loss can be kept constant between range sets by maintaining a constant distance between consecutive focusing ranges in inverse range (i.e., $\frac{1}{R_i} - \frac{1}{R_{i+1}} = $ constant). This distance, however, while designed to remain constant across ranges, must also be made to change as a function of $\sin^2(\text{MRA})$, for the reasons explained above.

Utilizing this information, a particular formula was designed as follows:

$$FR = \left\{ \begin{array}{ll} R_{\max} \frac{\sin^2(\text{MRA})}{N_{\text{FR}} - r} & 45° \leq \text{MRA} \leq 135° \\ R_{\max} \frac{\sin^2(45)}{N_{\text{FR}} - r} & \text{MRA} < 45° \\ R_{\max} \frac{\sin^2(135)}{N_{\text{FR}} - r} & \text{MRA} > 135° \end{array} \right\} r = 1 \text{ (closest)}, 2, \cdots, N_{\text{FR}} \text{ (farthest)}, \tag{17}$$

where N_{FR} is the number of focusing range sets desired and $R_{\max}$ is the maximum value of focusing range allowed for all MRA and r, where r is the range set index. Equation (17) is illustrated in figure 2 for $R_{\max} = 18,000$ yards and $N_{\text{FR}} = 5$. An infinite number of realizations is possible. If the spacing between consecutive focusing ranges is kept constant, there remains one degree of freedom, which is the location of one of the focusing ranges.
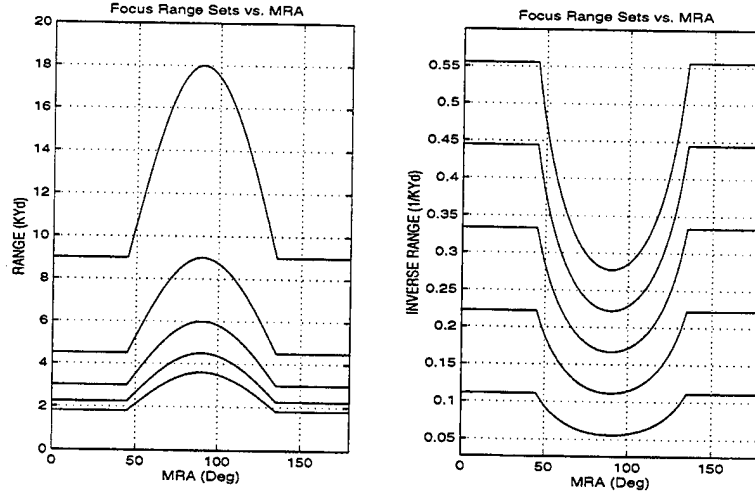
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Figure 2. *Six Sets of Focusing Ranges as a Function of Beamformer MRA, Expressed in Range and Inverse Range*

## 4.3   USING K-ω VALUES OUTSIDE THE ACOUSTIC CONE

To perform the focusing and/or interpolation, points outside the acoustic cone (waves flowing below the speed of sound) must be used. Such points will produce a $\cos(\theta)$ in the $k \to \theta$ equation that is greater than one in absolute value. This can be dealt with, since in the equations needed to obtain the coefficients, such as the time-delay and the range-vs-bearing equations, only the $\sin^2(\theta)$ is needed, which can be replaced by $1 - \cos^2(\theta)$.

4.4 K-ω APPLICATION

The K-ω field obtained by the two-dimensional FFT method briefly described above is the *infinite-range* K-ω. For a given frequency (bin i_0) and look direction θ_0 (bin $k_0 = M_{\text{fft}} d \cos \theta_0 / \lambda$), it is of interest to use different neighboring points (beams) along the k-axis (i.e., differently steered beams focused at infinity) to produce a differently focused beam at the same point (frequency and steering bearing) of interest. So, for K-ω point (i_0, k_0), it is of interest to use the L infinitely focused points (i_0, k_0), $(i_0, k_0 \pm 1)$, $(i_0, k_0 \pm 2)$,..., selected around (i_0, k_0), but not necessarily symmetric in k about k_0, in an interpolation scheme like the one described above, to replace the original infinitely focused point at (i_0, k_0) with a new one focused at a prescribed range.

The interpolation scheme derived in section 3 was implemented for different values of L for a long line array with equally spaced elements. Target frequencies of $F_s/4$ and $F_s/16$, where $F_s = 3.22$ of the array design frequency and MRA location $\approx 61°$, were selected so that the target is located exactly on a K-ω bin. The resultant beampatterns – exact versus interpolated – are overlaid, showing the accuracy of the scheme as a function of the number

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of interpolation points. The target was swept in bearing at the MRA-varying range defined by equation (17) for the two extreme range-focusing cases of $r = 6$ (farthest) and $r = 1$ (closest), with $N_{FR} = 6$, as shown in figure 2. The exact-beampattern's focusing range was identical to target range in each case. The technique is illustrated in figure 3. Simulation results for different numbers of interpolation coefficients and frequencies are shown in figures 4-11.
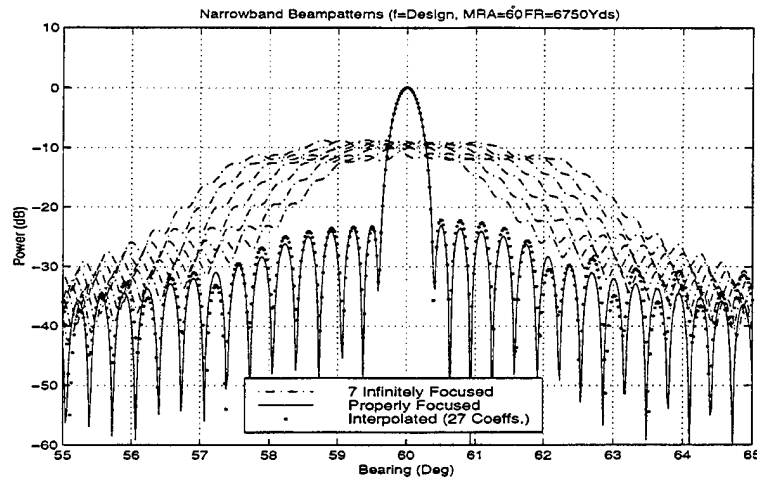
Figure 3. Illustration of MSE Minimization Method for Interpolating 27 Beams Equi-Spaced in Wavenumber Space and Infinitely Focused to a Range Matching the Target's at 6750 Yards
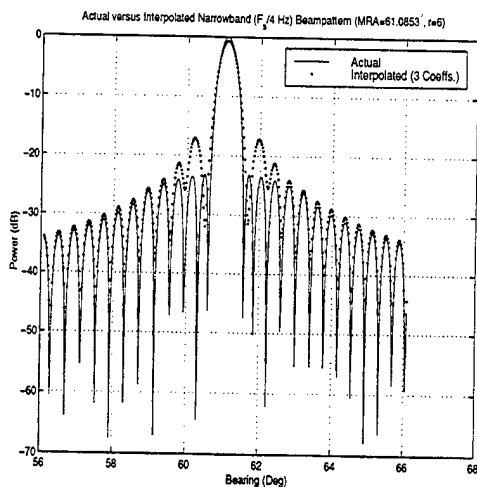
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Figure 4. **Time-Delay Focused (r = 6, N_{FR} = 6) Narrowband Beampattern (f = $F_s/4$) vs Interpolation of Three Infinitely Focused Ones**

Figure 5. **Time-Delay Focused (r = 6, N_{FR} = 6) Narrowband Beampattern (f = $F_s/4$) vs Interpolation of Seven Infinitely Focused Ones**

Figure 6. **Time-Delay Focused (r = 6, N_{FR} = 6) Narrowband Beampattern (f = $F_s/16$) vs Interpolation of Three Infinitely Focused Ones**
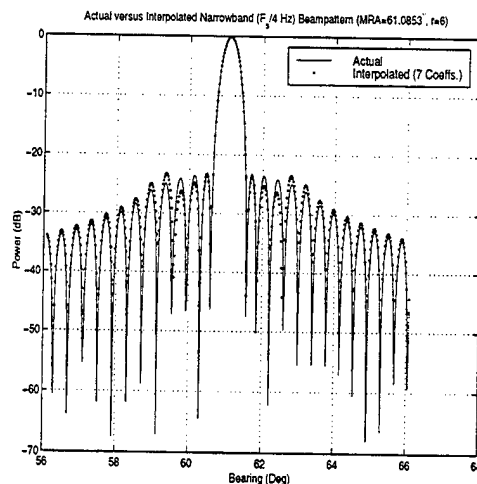

Figure 7. **Time-Delay Focused (r = 6, N_{FR} = 6) Narrowband Beampattern (f = $F_s/16$) vs Interpolation of Seven Infinitely Focused Ones**
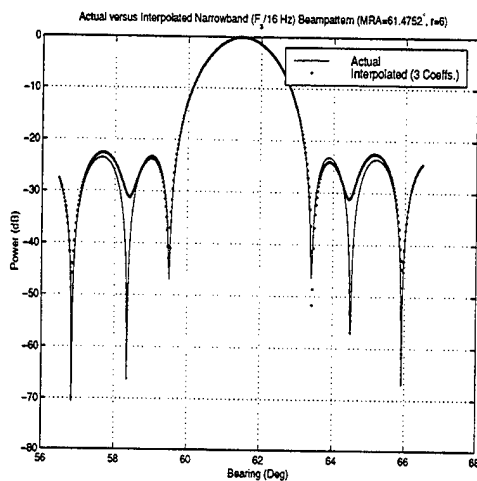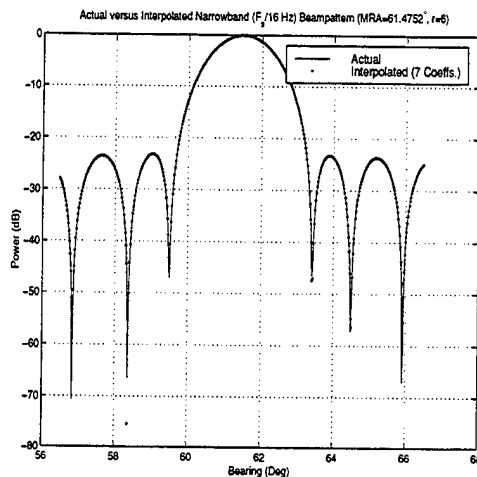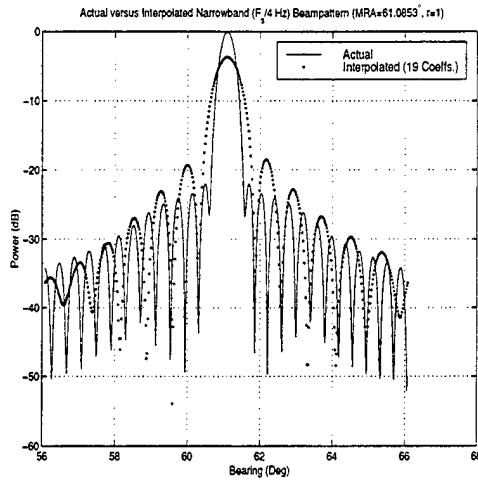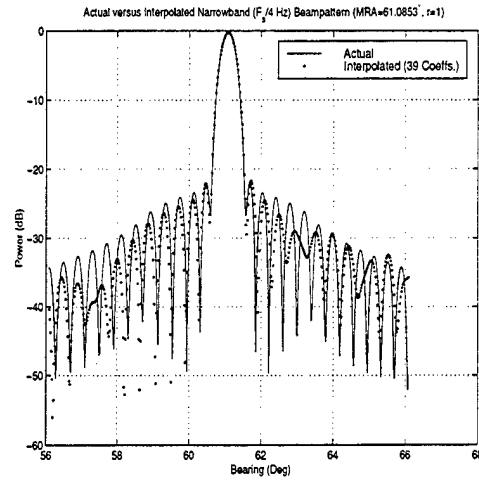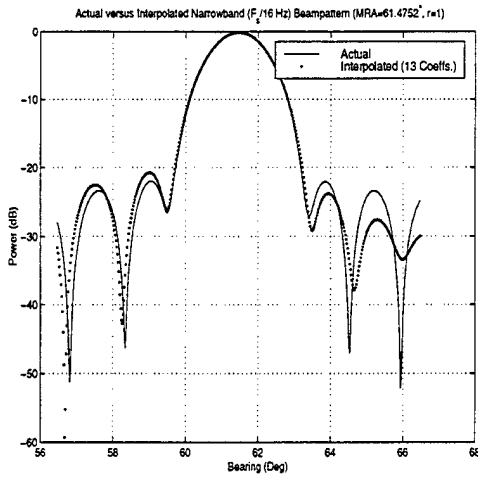
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Figure 8. *Time-Delay Focused (r = 1, N_{FR} = 6) Narrowband Beampattern (f = $F_s/4$) vs Interpolation of 19 Infinitely Focused Ones*

Figure 9. *Time-Delay Focused (r = 1, N_{FR} = 6) Narrowband Beampattern (f = $F_s/4$) vs Interpolation of 39 Infinitely Focused Ones*

Figure 10. *Time-Delay Focused (r = 1, N_{FR} = 6) Narrowband Beampattern (f = $F_s/16$) vs Interpolation of 13 Infinitely Focused Ones*
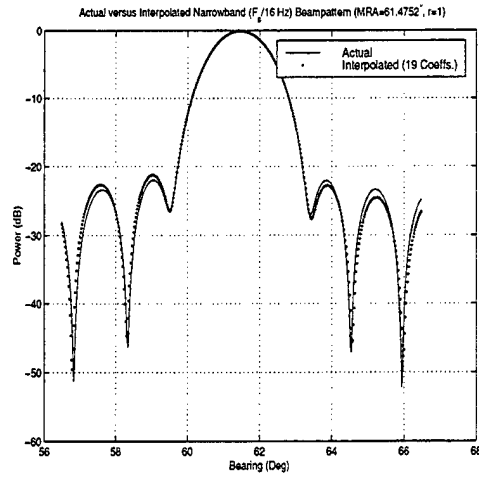
Figure 11. *Time-Delay Focused (r = 1, N_{FR} = 6) Narrowband Beampattern (f = $F_s/16$) vs Interpolation of 19 Infinitely Focused Ones*

The error between the interpolated infinitely focused beampatterns and the actual beampattern, given by equation (7), is computed using the optimal values of α and β for different numbers of coefficients L and different desired focusing ranges R_0. The result is plotted for

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two different target frequencies ($F_s/4$ and $F_s/16$) in figures 12 and 13, respectively.
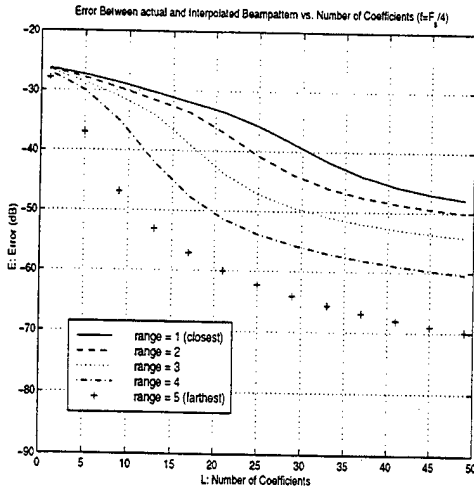
Figure 12. *Error Between Actual and Interpolated Beampatterns (Infinitely Focused) vs Number of Interpolation Coefficients for Six Focusing Ranges at Frequency $F_s/4$*
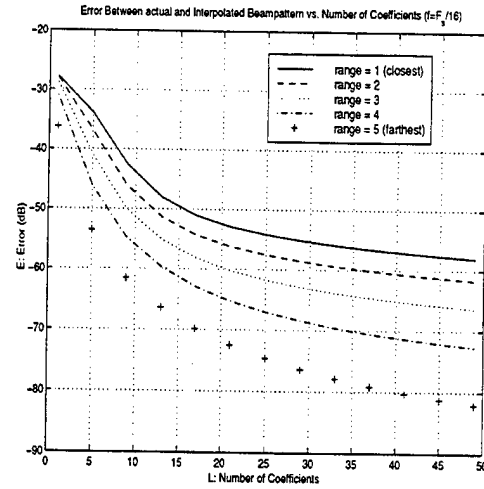
Figure 13. *Error Between Actual and Interpolated Beampatterns (Infinitely Focused) vs Number of Interpolation Coefficients for Six Focusing Ranges at Frequency $F_s/16$*

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5. CONCLUSIONS

As expected, the larger the number of coefficients used to interpolate, the better the resultant beam approximation. In addition, it is clear from figures 4-13 that a lower number of interpolation coefficients is required at lower frequencies (under similar circumstances) and vice versa. Furthermore, as shown in figures 12 and 13, the number of interpolation coefficients also increased as the focusing range became smaller. These observations prompt the need to use a different number of coefficients for different frequency bands. Furthermore, rather than using infinitely focused beams to interpolate to the other N_{FR} ranges, it would be more advantageous to use beams focused at range $N_{FR}/2$ (N_{FR} even) and then interpolate up/down to the farther/closer desired focusing ranges. As expected, however, if the focused beams formed at range $N_{FR}/2$ were originally formed from infinite focus range ones, then errors in those interpolations will be compounded in further interpolations. It is important to note that the error computed in equation (7) is not the only criterion needed to determine the number of coefficients. The beampatterns, with their sidelobe level and mainlobe width, must also be examined.

It must be noted also that alternatives to the mean square error minimization method for beam interpolation exist. The convolution method employed by Bernecky [2], for example, while not as accurate as the MSE minimization method, is computationally much faster.

REFERENCES

[1] Louis Scharf, *Statistical Signal Processing,* Addison-Wesley, Reading, MA, 1991, p. 274.

[2] W. Robert Bernecky, "*Range Focused K-ω Beamforming of a Line Array,*" NUWC-NPT Technical Memorandum 980118, Naval Undersea Warfare Center Division, Newport, RI, 25 September 1998.

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APPENDIX A
VECTOR AND MATRIX DEFINITION FOR
NON-SHADING-INCLUDED COEFFICIENTS

Sometimes, it is of interest to obtain focusing-steering coefficients that don't include the shading weights. It can be shown that this can be simply implemented by using a different definition than (5) and (6) for the vectors $\mathbf{c}$ and $\mathbf{s}$ and the matrices $\mathbf{C}$ and $\mathbf{S}$. These are stated here without derivation:

$$\mathbf{c} = \left[\sum_{i=1}^{N} w_i^2 \cos v_i(1), \sum_{i=1}^{N} w_i^2 \cos v_i(2), \cdots, \sum_{i=1}^{N} w_i^2 \cos v_i(L)\right]^{T},$$

$$\mathbf{s} = \left[\sum_{i=1}^{N} w_i^2 \sin v_i(1), \sum_{i=1}^{N} w_i^2 \sin v_i(2), \cdots, \sum_{i=1}^{N} w_i^2 \sin v_i(L)\right]^{T},$$

$$[\mathbf{C}]_{pq} = \sum_{i=1}^{N} w_i^2 \cos[v_i(q) - v_i(p)] = [\mathbf{C}]_{qp},$$

$$[\mathbf{S}]_{pq} = \sum_{i=1}^{N} w_i^2 \sin[v_i(q) - v_i(p)] = -[\mathbf{S}]_{qp}.$$

The formula used to compute the coefficients is equation (9) in the main text.

APPENDIX B
MATLAB CODE FOR COEFFICIENT COMPUTATIONS

```
% MATLAB CODE TO OBTAIN BEAM/K INTERPOLATION/FOCUSING COEFFICIENTS
%
% AUTHOR: Sami Deeb, CODE 2123


function coefficients=gen_coeff(N,M_fft,M,f,d,c,R0,R,cos_th0,cos_th,shad_embded)


% FUNCTION WHICH USES M AVAILABLE LINE-ARRAY SINGLE-FREQUENCY BEAMS, STEERED TO
% GIVEN DIRECTIONS, AND FOCUSED AT GIVEN RANGES, MEASURED AT ONE FREQUENCY
% VALUE, TO PRODUCE A NEW BEAM STEERED TO A DESIRED DIRECTION AND FOCUSED AT
% A DESIRED RANGE.
%
% N      NUMBER OF ARRAY ELEMENTS
% M_fft  THE SMALLEST RADIX-2 NUMBER THAT IS LARGER THAN THE NUMBER OF ELEMENTS
% f      BEAM/BEAMPATTERN FREQUENCY
% d      ELEMENT SEPARATION
% c      SPEED OF SOUND
% M      NUMBER OF COEFFICIENTS (ODD)
% cos_th 1xM VECTOR DENOTING THE COSINE OF ANGLES WHERE THE AVAILABLE
%        BEAMS ARE STEERED TO.
% cos_th0 COSINE OF ANGLE OF DESIRED BEAM
% R      1xM VECTOR REPRESENTING THE RANGE WHERE THE AVAILABLE BEAMS HAVE
%        BEEN FOCUSED TO.
% R0     RANGE OF DESIRED BEAM
% shad_embded DENOTES WHETHER THE GENERATED COEFFICIENTS SHOULD INCLUDE THE
%        SHADING OPERATION (shad_embded='y') OR NOT (shad_embded='n')
%
% sphr_td(element_coordinate,cos(th),R,c)  A FUNCTION THAT COMPUTES THE TIME
%        DELAY OF A WAVE FRONT TRAVELING AT SPEED c, INCIDENT ON THE ARRAY AT
%        AN ANGLE WHOSE COSINE IS cos_th, AND FROM A RANGE R, AT AN ARRAY
%        ELEMENT, DEFINED BY ITS COORDINATES element_coordinate.


shade    = taylor(N,23,6);                    % TAYLOR SHADING SL=23 dB, nbar = 6
shade    = shade(:)'/sum(shade);              % CONVERT TO ROW/NORMALIZE ~ 1xN
zero_pad = zeros(1,(M_fft-N)/2);              % LEFT AND RIGHT ZERO-PADDING ARRAY
shade    = [zero_pad,shade,zero_pad];         % ZERO-PADDED 1xM_fft TAYLOR ARRAY
uniform  = [zero_pad,ones(1,N),zero_pad];     % ZERO-PADDED 1xM_fft UNIFORM ARRAY
shade_sq = shade.^2;                          % 1xM_fft


if (shad_embded == 'n') CC_SS = shade_sq; cc_ss = shade_sq;   %CC_SS: 1xM_fft
```

```
else                           CC_SS = uniform;    cc_ss = shade;           %cc_ss: 1xM_fft
end


elmnt   = d*[0:M_fft-1]-(M_fft-1)*d/2;        %COMPUTING ELEMENT COORDINATES


V        = zeros(M_fft,M);
tmp_CC   = zeros(M_fft,M,M);      tmp_SS = zeros(M_fft,M,M);
CC       = zeros(M,M);            SS     = zeros(M,M);
col_ones = ones(M,1);


for m=1:M_fft   %ELEMENT TIME DELAY/PHASE COMPUTATION LOOP
    V(m,:)=2*pi*f*(sphr_td(elmnt(m),cos_th0,R0,c)- ...
                   sphr_td(elmnt(m),cos_th,R,c));
end


for m=1:M_fft   %ELEMENT LOOP
   temp            = col_ones*V(m,:);          %temp : MxM MATRIX OF IDENTICAL ROWS
   temp2           = temp - temp';             %temp2 = V(m,j) - V(m,i)
   tmp_CC(m,:,:) = CC_SS(m)*cos(temp2);
   tmp_SS(m,:,:) = CC_SS(m)*sin(temp2);
end


% SUMMING tmp_CC AND tmp_SS OVER THE FIRST DIMENSION REPRESENTING ELEMENTS ----
CC   = reshape(sum(tmp_CC),M,M);        SS = reshape(sum(tmp_SS),M,M);
cc   = (cc_ss*cos(V))';                 ss = (cc_ss*sin(V))';          %ss & cc: Mx1

inv_CC = CC\eye(size(CC));            %inv_CC = inv(CC);
a=(CC+SS*(CC\SS))\(cc-SS*(CC\ss));    %a= inv(CC+SS*inv_CC*SS)*(cc-SS*inv_CC*ss);
b= -CC\(ss+SS*alpha);                 %b= - inv_CC*(ss+SS*alpha);


coefficients = a + i*b;       %Mx1
```

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